Existence Results for abstract Impulsive semilinear differential Equations

H. Hammouche, K. Guerbati

Laboratoire de Mathématiques et des Sciences appliquées
Université de Ghardaia.
e-mail: h.hammouche@yahoo.fr, guerbati_k@yahoo.com.

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1. Introduction

In this paper, we shall be concerned with existence of mild solution solutions defined on a compact real interval for first order impulsive semilinear functional inclusions with state-dependent delay in a separable Banach space of the form:

$$y'(t) - Ay(t) = f(t, y_t), \quad t \in J = [0, b], \tag{1}$$

$$\Delta y(t_i) \in I_k(y_{t_k}), \quad k = 1, 2, \dots, m, , \tag{2}$$

$$y(t) = \phi(t), \quad t \in (-r, 0], \tag{3}$$

where $f : J \times \mathcal{D} \to E$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \to E, \psi$ is continuous every where except for a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in D$, $0 < r < \infty$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $I_k \in C(E, E)$, $k = 1, 2, \dots, m$, $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, and E a real separable Banach space with norm $|.|$.

We denote by y_t the element of D defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from $t - r$, up to the present time t.

2. Preliminaries

In this section, we introduce notations and preliminary facts that are used throughout this paper. $C(J, E)$ is the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and $\forall r > 0$ B_r denotes the closed ball in E centered in 0_E, with radius r.
Here $\rightharpoonup$ weak convergence and $\rightarrow$denotes strong convergence in E, respectively.
$B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(J, E)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| dt.$$

$\mathcal{B}(E)$ denotes the collection of all nonempty subsets of E. $\mathcal{W}(E)$ is the subset of $\mathcal{B}(E)$consisting of weakly compact subsets of E.

Definition 2.1. *The measure of weak noncompactness is a map* $\omega : \mathcal{B}(E) \longrightarrow [0, +\infty[$ *defined by:*

$$\omega(M) = \inf\{r > 0, \exists w \in \mathcal{W}(E) with M \subseteq w + B_r\}$$

foe every $M \in \mathcal{B}(E)$

The measure of weak noncompactness was introduced by Blasi[5].It has the following properties:see Appell and Pascale [4, 5]
Let M_1, M_2 be two elements of $\mathcal{B}(E)$. the following properties hold:

1. If $M_1 \subseteq M_2$ then $\omega(M_1) \leq \omega(M_2)$
2. $\omega(M_1) = 0$ if and only if $\overline{M_1}^w \in \mathcal{W}(E)$ ($\overline{M_1}^w$ means the weak closure of M_1)
3. $\omega(\overline{M_1}^w) = \omega(M_1)$
4. $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$
5. $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ for all $\lambda \in \mathbb{R}$
6. $\omega(co(M_1)) = \omega(M_1)$
7. $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$

In L^1-spaces the map $\omega(.)$ can be expressed as

$$\omega(M) = \limsup_{\epsilon \to 0} \left\{ \sup_{\Psi \in M} \left[\int_D ||\Psi(t)||_X dt; |D| \leq \epsilon \right] \right\}$$

for every bounded subset M of $L^1(\Omega; X)$ where X is a finite dimensional Banach space and $|D|$ is the Lebesgue measure of the set D. Now, we introduce the concept of separate contraction mapping which was introduced in [7]

Definition 2.2. *Let X be a Banach space and $f : X \longrightarrow X$ is said to be a separate contraction mapping if there exist two functions $\varphi, \psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying:*

(a) $\psi(0) = 0$, *ψ is strictly increasing,*
(b) *φ is continuous,*
(c) $||f(x) - f(y)|| \leq \varphi(||x - y||)$,
(d) $\psi(r) \leq r - \varphi(r)$ *for* $r > 0$.

We will use the following two conditions:

($\mathcal{A}_1$): If $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ is a weakly convergent sequence in X, then $T(x_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

$(\mathcal{A}_2)$: If $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ is a weakly convergent sequence in X, then $T(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

Definition 2.3. *The superposition operator generated by a carathéodory function f is the mapping defined by:*

$$\begin{array}{rccc} N_f : & L^1(\Omega) & \longrightarrow & L^1(\Omega) \\ & y(t) & \longmapsto & N_f(y(t)) = f(t, y(t)), \end{array}$$

where $L^1(\Omega)$ is the set of all mesurable functions on Ω.

Lemma 2.1. *see* [6] *and the references therein Let $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ be a carathéodory function. Then the superposition operator N_f maps $L^1(\Omega)$ into itself if and only if there exist a constant $b \geq 0$ and a function $a(\cdot) \in L^1_+(\Omega)$ such that:*

$$|f(t, y)| \leq a(t) + b|y|$$

where $L^1_+(\Omega)$ denotes the positive cone of the space $L^1(\Omega)$.

Lemma 2.2. *Let Ω be a boubded domain in $\mathbb{R}^n$. If $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ is a carathéodory function and N_f maps $L^1(\Omega)$ into itself, then N_f satisfies $(\mathcal{A}_2)$*

The following fixed point theorem is crucial for our purposes.

Theorem 2.1. [6] *Let X be a banach space and $A, B : X \longrightarrow X$ be two continuous mapping. If A, B satisfy the following conditions,*

(i) *A maps bounded sets into relatively weakly compact ones,*

(ii) *A satisfies $(\mathcal{A}_1)$,*

(iii) *B is a separate contraction satisfying condtion $(\mathcal{A}_2)$,*
Then , either

(a) *the equation $x = Ax + Bx$ has a solution, or*

(b) *The set $\mathcal{E} = \left\{x \in X : x = \lambda B(\frac{x}{\lambda}) + \lambda A(x)\right\}$ is unbounded for $\lambda \in (0, 1)$.*

3. EXISTENCE RESULTS

we shall consider the space

$$\begin{aligned} PC \;\; = \Big\{ & y : [0, b] \to D(A) : y_k \in C(J_k, D(A)), k = 0, \dots, m \quad \text{such that} \\ & y(t_k^-), y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-), k = 1, \dots, m \Big\} \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty, k = 1, \dots, m\}$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}], k = 0, \dots, m$.

Set

$$\Omega = \{y : [-r, b] \to D(A) : y \in D \cap PC\}.$$

Then Ω is a Banach space with norm

$$\|y\|_\Omega = \max(\|y\|_D, \|y\|_{PC}).$$

In this section, we give our main existence result for problem (1)–(3). Before stating and proving this result, we give the definition of its mild solution.

Definition 3.1. *A function $y \in PC([-r, b], E)$ is said to be a mild solution of problem (1)–(3) if $y(t) = \phi(t)$, $t \in [-r, 0]$, and y is a solution of impulsive integral equation*

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)f(t, y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad t \in J.$$

Let us introduce the following hypotheses:

(H1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a semi-group $\{T(t)\}$, $t \in J$. Let $M = \sup\{\|T(t)\|_{B(E)} : \ t \in J\}$;

(H2) There exist constants $d_k > 0,\ k = 1, ..., m$ with $M \sum_{k=1}^{m} d_k < 1$ such that for each $y, x \in E$

$$|I_k(y) - I_k(x)| \le d_k |y - x|$$

(H3) The function $f : J \times D \to E$ is Carathéodory and there exist a constant $\rho > 0$ and a function $p \in L^1(J, \mathbb{R}_+)$ such that:
$|f(t, x)| \le p(t) + \rho|x|$

Theorem 3.1. *Under assumtions (H1), (H2) and (H3), the pronlem 1-3 has at least one mild solution*

Proof. Transform the problem (1)-(3) into a fixed point problem. Consider the two operators:

$$\mathcal{A}, \mathcal{B} : L^1(\Omega) \to L^1(\Omega)$$

defined by

$$\mathcal{A}(y)(t) := \begin{cases} 0, & \text{if } t \in [-r, 0]; \\ \displaystyle\sum_{0<t_k<t} T(t-t_k) I_k\left(y\left(t_k^-\right)\right), & \text{if } t \in J, \end{cases}$$

and

$$\mathcal{B}(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r, 0]; \\ \displaystyle T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)\, ds, & \text{if } t \in J. \end{cases}$$

Then the problem of finding the solution of problem (1)–(3) is reduced to finding the solution of the operator equation $\mathcal{A}(y)(t) + B(y)(t) = y(t), t \in [-r, b]$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfies all the conditions of Theorem 2.1

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