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Existence Results for Impulsive Semilinear Fractional Differential Inclusions with Delay in Banach spaces

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Abstract
In this work, we introduce a new concept of mild solution of some class of semilinear fractional differential inclusions of order $0 < \alpha < 1$. *Also we establish an existence result when the multivalued function has convex values. The result is obtained upon the nonlinear alternative of Leray-Schauder type.*

Résumé
Dans ce travail, nous introduisons une nouvelle dfinition de la solution faible de certaines inclusions différentielles fractionnaires semi-linéaires d'ordre $0 < \alpha < 1$. *De même nous établissons un résultat d'existence dans le cas où la fonction multivoque est à valeurs convexes. Le résultat est obtenu moyennant l'alternative non linéaire de type Leray-Schauder.*

1 Introduction

Our aim in this paper is to study the existence of mild solutions for fractional semilinear differential inclusions of the form:

$$^{c}D^{\alpha}_{t_k}y(t) - Ay(t) \in F(t, y_t),\ t \in J_k := (t_k, t_{k+1}],\ k = 0, \ldots, m \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\quad k = 1, \ldots, m \tag{1.2}$$

Key Words and Phrases:
Mots Clés et Phrases: *calcul fractionnaire; dérivée fractionnaire de Caputo; transformation multivoque; solution faible; point fixe*

$$y(t) = \phi(t) \qquad t \in [-r, 0], \tag{1.3}$$

where $0 < \alpha < 1$, $F : [0, b] \times D \to \mathcal{P}(E)$ is a given valued multivalued map, $I_k : E \to E$, , $A : D(A) \subset E \to E$ is the generator of an $\alpha-$resolvent operator function ($\alpha-$ROF for short) S_α. Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering ([2]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see for example the monographs of Abbas *et al.* [1]. The Cauchy problem for abstract differential equations involving Riemann-Liouville fractional integral have been considered by several author; see for example Benchohra *et al.* [3] and the reference therein.

2 Main results

We consider the following space

$$PC = \Big\{ y : [0, b] \to E : y \in C((t_k, t_{k+1}], E);\ k = 0, \dots, m \quad \text{such that} \quad y(t_k^-),\ y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-),\ k = 1, \dots, m \Big\}$$

which is a Banach space with the norm

$$\|y\|_{PC} := \max\{\|y_k\|_\infty : k = 0, \dots, m\}$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}],\ k = 0, \dots, m$
Set

$$\Omega = \{y : [-r, b] \to E : y \in D \cap PC\}.$$

Now, we can define a meaning of the mild solution of problem (1.1)-(1.3).

Definition 2.1. A function $y \in \Omega$ is called to be mild solution of (1.1)-(1.3) if $y(t) = \phi(t)$ for all $t \in [-r, 0]$, $\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m$ and there exists $v(\cdot) \in L^1(J, E)$, such that $v(t) \in F(t, y_t)$, a.e. $t \in [0, b]$, and such that y satisfies the following integral equation:

$$y(t) = \begin{cases} S_\alpha(t)\phi(0) + \displaystyle\int_0^t S_\alpha(t-s)v(s)ds & \text{if } \ t \in [0, t_1], \\ S_\alpha(t-t_k)\displaystyle\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) & \\ + \displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)\times & \\ S_\alpha(t_i - s)v(s)ds + \displaystyle\int_{t_k}^{t} S_\alpha(t-s)v(s)ds & \\ + \displaystyle\sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)I_i(y(t_i^-)), & \text{if } \ t \in (t_k, t_{k+1}]. \end{cases} \tag{4}$$

Let us introduce the following hypotheses:

(H1) A generates a compact $\alpha-$ROF S_α for $t > 0$ which is exponentially bounded, i.e., there exist constants $M \geq 1, \omega \geq 0$ such that:

$$\|s_\alpha(t)\| \leq Me^{\omega t}, \qquad t \geq 0.$$

(H2) $F : J \times D \to \mathcal{P}_{cp,cv}(E)$ is Carathéodory and there exist $p \in L^\infty(J, \mathbb{R})$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|_\infty) \quad \text{for all} \;\; t \in J, \; u \in D,$$

with

$$\int_{C_3}^{\infty} \frac{du}{\psi(u)} = \infty,$$

(H3) The functions $I_k : E \to E$ are continuous and there exists a constant $M^* > 0$ such that

$$|I_k(u)| \leq M^* \text{ for each } u \in E, \; k = 1, \ldots, m.$$

Theorem 2.1. *Under assumptions (H1)-(H3) the IVP (1.1)-(1.3) has at least one mild solution on* $[-r, b]$.

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the multivalued operator: $N : \Omega \to \mathcal{P}(\Omega)$ defined by $N(y) = \{h \in \Omega\}$ such that

$$h(t) = \begin{cases} \phi(t), & \text{if} \;\; t \in [-r, 0], \\ S_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)v(s)ds & \text{if} \;\; t \in [0, t_1], \\ S_\alpha(t - t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})\phi(0) & \\ +\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t - t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j)\times & \\ S_\alpha(t_i - s)v(s)ds + \int_{t_k}^{t} S_\alpha(t-s)v(s)ds & \\ +\sum_{i=1}^{k} S_\alpha(t - t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)), & \text{if} \;\; t \in (t_k, t_{k+1}]. \end{cases}$$

3 Conclusion

To our knowledge, there are very few results for impulsive fractional differential inclusions. The results of the present work extend and complement those obtained in the absence of the impulse functions I_k.

References

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